Unit 6 Sequences and Series

Sequences:

A sequence is an ordered list of numbers that follow a specific pattern. The sequence may stop or may continue on indefinitely

Ex) 5, 7, 9, 11, 13, … Pattern: +2 each time

7, 21, 63, 189, 567, … Pattern.

100, 50, 25, 12.5, … Pattern:

Notation:

The terms within a sequence are named in order using the notation $t_1,\ t_2,\ t_3,$ etc.

Ex)

1,	4,	9,	16,	25,	36, …
t_1	t_2	t_3	t_4	t_5	t_6

Ex) 3, 7, 11, 15, 19, 23, 27, 31, 35, …

$t_4 =$ ____ $t_7 =$ ____

Representing Sequences:

To represent a sequence we can simply list out the first few terms

this can continue

Ex) 4, 20, 100, 500, … FOREVER...

or we can use a formula that describes how each term is generated.

General Term: * This helps us figure out terms without having to list out the entire pattern.

The general term is a formula used to describe a sequence that is based on the term number. (If you want to know the value of the 5th term you put a 5 into the formula.)

Ex) $t_n = 3n - 2$, find the first 3 terms

Term 1: n=1 Find t_2 and t_3.

$t_1 = 3(1) - 2$

$t_1 = 3 - 2$

$\boxed{t_1 = 1}$

$t_n = n^2 + 1$, find the 23rd term $\therefore n = 23$

If $t_n = 7n - 5$, which term is equal to 51?

This means that some term in our sequence is 51 and we want to know which term ($t_1, t_2, t_3, \ldots$?). We want to figure out n.

$51 = 7n - 5$ Solve for n.

Types of Sequences:

There are two main types of sequences we will consider in this course they are:

Arithmetic Sequence
(addition/subtraction pattern)
Is a sequence that increases (or decreases) by a $\boxed{\text{common difference.}}$

+6 +6 +6

Ex) 7, 13, 19, 25, …

Geometric Sequence
(multiplication/division pattern)
Is a sequence that increases (or decreases) by a common ratio.

×3 ×3 ×3

Ex) 2, 6, 18, 54, …

Arithmetic Sequences:

in a row

Is a sequence in which the difference between consecutive terms is a constant. This constant is called the common difference.

+2 +2 +2 +2 +2

Ex) 3, 5, 7, 9, 11, 13, … common difference: 2

-5, -1, 3, 7, 11, … common difference:

2, 9, 16, 23, 30, … common difference:

The general term of an arithmetic sequence can be found using the formula:

1st term

difference between terms.

$$\boxed{t_n = a + (n-1)d}$$

a = value of the first term

d = common difference between terms

Ex) Find the general term of the following sequences.

a) 8, 12, 16, … (+4, +4)

$t_n = a + (n-1)d$

$t_n = 8 + (n-1)(4)$

$t_n = 8 + 4n - 4$

$\boxed{t_n = 4n + 4}$ ← This can now be used to find any term.

b) 3, 8, 13, 18, …

c) 23, 19, 15, 11, …

Ex) Determine the indicated term for each of the following.

* These terms are BIG! Find your general Formula first *

a) 8, 11, 14, 17, … (+3, +3, +3) find t_{31}

$t_n = a + (n-1)d$

$t_n = 8 + (n-1)(3)$

$t_n = 8 + 3n - 3$

$\boxed{t_n = 3n+5}$

* Use to find t_{31} *

$t_{31} = 3(31) + 5$

$t_{31} = 93 + 5$

$\boxed{t_{31} = 98}$

b) 37, 30, 23, 16, … find t_{74}

* What is different about d in this case? *

Find general formula.

Find t_{74}.

Ex) Determine how many terms are indicated by each of the following sequences. We want to know the term #, n, for 442.

a) -20, -9, 2, 13, … …, 442

(t_1, t_2, t_3, t_4 … $t_?$; +11, +11, +11)

Set up general formula

$t_n = a + (n-1)d$

$t_n = -20 + (n-1)(11)$

$t_n = -20 + 11n - 11$

$\boxed{t_n = 11n - 31}$

We have our general Formula and we know $t_n = 442$ so . .

$t_n = 11n - 31$

$442 = 11n - 31$ Solve for n.

$442 + 31 = 11n \cancel{-31} \cancel{+31}$

$\frac{473}{11} = \frac{\cancel{11}n}{\cancel{11}}$

$\boxed{n = 43}$

b) 31, 36, 41, … …, 791 Try ME!

Arithmetic Means:

Arithmetic means are the terms between two non-consecutive terms on an arithmetic sequence.

Ex) 2, 5, 8, 11, 14, 17, 20, 23

The arithmetic means between 2 and 23 are
5, 8, 11, 14, 17, 20 because these all fall between 2 & 23

The arithmetic means between 5 and 17 are?

____, ____, ____

Ex) Find the $\boxed{\text{four}}$ arithmetic means between 27 and 47.
We know there will be 4 numbers between 27 & 47 so...

27, ____, ____, ____, ____, 47
+? +? +? +?

$t_n = a + (n-1)d$

$47 = 27 + (6-1)d$

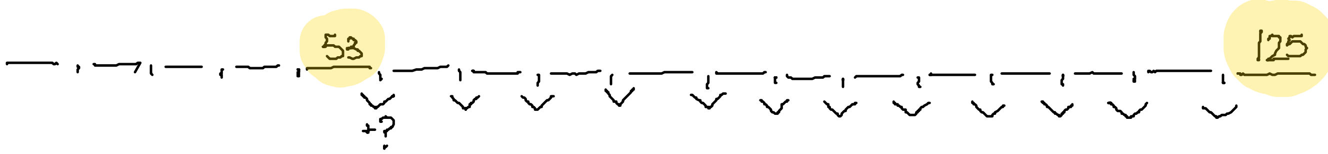
$47 = 27 + 5d$ Solve for d. If you know d you can either:

a) Set up general formula & solve for t_2, t_3, t_4, t_5

b) Count up in your pattern by d.

Ex) Determine the general term for the arithmetic sequence that has $t_5 = 53$ and $t_{17} = 125$ as two of its terms.

5^{th} term $\qquad$ 17^{th} term



What is d between each term? If we find d then we can use this to find a.

General $\qquad$ We need these.

$t_n = a + (n-1)d$

Now Try
Page 16 #1 to 9, 10, 11, 13, 16, 17

Arithmetic Series:

When the terms of a sequence are added together we call it a series.

Ex) 3, 5, 7, 9, 11, 13, …. is an arithmetic sequence

3 + 5 + 7 + 9 + 11 + 13 + … is an arithmetic series

Notation:

The partial sums of a series are denoted using the following notation:

S_5 means sum of the first 5 terms

S_{31} means sum of the first 31 terms

Ex) Given the series below, find the following partial sums.

3 + 5 + 7 + 9 + 11 + 13 + …

a) S_3 b) S_7 c) S_{79}

Ex) What is the sum of the first 100 natural numbers?

From this thinking we can develop a formula to determine the sum of an arithmetic series.

$$\boxed{S_n = \frac{n}{2}\left(a + t_n\right)}$$

But $t_n = a + (n-1)d$ so

$$S_n = \frac{n}{2}\left[a + a + (n-1)d\right]$$

$$\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]}$$

Ex) Find the sum of the first 70 terms of the series given below.

$8 + 15 + 22 + ...$

Ex) Determine S_{37} for the series $4 + 10 + 16 + ...$

Ex) Determine the sum of each series given below.

a) $17 + 30 + 43 + 56 + \ldots. \qquad \ldots + 563$

b) $249 + 243 + 237 + \ldots \qquad + 27$

Ex) Determine S_{11} for the arithmetic series that has $t_1 = -9$ and $t_{11} = 31$ as two of its terms.

Ex) If $t_1 = 25$, $t_n = 382$, and $S_n = 4477$ for a particular series, determine the number of terms *n* represented in the partial sum.

Ex) The sum of the first two terms of an arithmetic series is 13 and the sum of the first four terms is 46. Determine the first six terms of the series and the sum to six terms.

Now Try
Page 27 #1 to 7, 10, 11, 15, 16, 20

<u>Geometric Sequences:</u>

Geometric Sequences are sequences where there is a common ratio between successive terms.

Ex) 1, 3, 9, 27, 81, …
×3 ×3 ×3 ×3
Common ratio $= r =$ 3

Ex) $3\sqrt{2},\ 6,\ 6\sqrt{2},\ \ldots$
×? ×?

$r =$

What if...

1, 0.5, 0.25, 0.125, ...

r = ?

The explicit formula for a geometric sequence can be found with the following formula:

$$\boxed{t_n = ar^{n-1}}$$

How?

$a =$ value of first term
$r =$ common ratio (what are you multiplying by)

Ex) Determine the explicit formula for the sequence given by 5, 10, 20, 40, 80, … Find the value of t_8.

×2 ×2 ×2 ×2

2^2

2^3

2^4

* Even though we have 5 terms, we only have 4 jumps. That's why our exponent is n−1. *

General Formula:

$t_n = ar^{n-1}$
$= 5(2)^{n-1}$

↑ cannot simplify

Find t_8: n=8

$t_8 = 5(2)^{8-1}$
$= 5(2)^7$
$= 5(128)$
$= 640$

Ex) How many terms are represented by the sequence

4, 20, 100, …   …, 39 062 500 ← there should be an extra 0…

Can we use a general formula to solve for n?

Ex) A car is worth $30 000 when first purchased. Each year the car depreciates 15%. How much will it be worth in 8 years? Hint: $t_1 = 0$ years, $t_2 = 1$ year, $t_3 = 2$ years, … what term do we want?

decreases by 15%…

r is how much remains

Ex) Determine the values of the 3 geometric means between 147 and 352947.

Hint: Could you have 2 r values?

147, ___, ___, ___, 352 947

×r ×r ×r ×r

We need r.

Ex) Determine the explicit formula for the geometric sequence where $t_3 = 36$ and $t_{10} = 78732$.

___ , ___ , 36 , ___ , ___ , ___ , ___ , ___ , ___ , 78732

xr xr xr xr xr xr xr xr xr

Steps: ① Find r.

② Find general form.

Now Try
Page 39 #1, 3 to 8, 10, 12, 23, 26

<u>Geometric Series:</u>

A Geometric Series is the sum of the terms of a geometric sequence.

Ex) 2, 8, 32, 128, … ← Geometric Sequence

$2+8+32+128$ ← Geometric Series

These are the same but your series is added

The Partial Sum of a geometric series can be found by:

given term # → $\boxed{S_n = \dfrac{a\left(r^{n}-1\right)}{r-1}}$ or $\boxed{S_n = \dfrac{r t_n - a}{r-1}}$ ← given last considered term

$S_n =$ sum of terms
$a =$ value of the first term
$r =$ common ratio
$n =$ number of terms considered
$t_n =$ value of last term considered

Ex) Find the sum of the first 12 terms for the sequence given by $1+4+16+64+\ldots$

term #, n

×4 ×4 ×4

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$= \frac{1(4^{12}-1)}{4-1}$$

$$= \frac{16\,777\,215}{3}$$

$$\boxed{= 5\ 592\ 405}$$

Ex) Determine the sum of the first 8 terms of the geometric series given below.

$5+15+45+...$

16400

Ex) Find the sum of each geometric series given below.

a) $3\overset{\times -2}{-6}\overset{\times -2}{+12}\overset{\times -2}{-24}+... \quad +49152-98304$

given t_n, not term #.

$S_n = \frac{rt_n - a}{r-1}$

$= \frac{(-2)(-98304)-3}{-2-1}$

$=$

Careful with +/-

b) $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+... \quad ...+\frac{1}{19683}$

$\div 3$ or $\times \frac{1}{3}$

$S_n = \frac{rt_n - a}{r-1}$

$= \frac{\left(\frac{1}{3}\right)\left(\frac{1}{19683}\right)-\frac{1}{3}}{\frac{1}{3}-1}$

Careful with fractions & calculator

Ex) In a Scrabble tournament there are 256 entries. In round one of the tournament each of the entries are paired up to play the winner of each match then goes on to the second round of play with the losers of each match being eliminated. This process is then repeated in round two and so on until only two players remain for the final match. For this tournament, how many matches will be played and in how many rounds will this take?

Sequence represents # of matches each round.

1^{st} round: $\frac{256 \text{ players}}{2 \text{ players}} = 128$ matches

2^{nd} round: $\frac{128 \text{ peoples}}{2 \text{ players}} = 64$ matches

$\underset{\substack{\uparrow \\ 1^{st} \\ \text{round}}}{\underline{128}}, \underset{\substack{\uparrow \\ 2^{nd}}}{\underline{64}}, 32, 16, 8, 4, 2, 1$

8 rounds

255 matches

Now Try
Page 53 #1 to 7, 9, 11, 16, 17

<u>Infinite Geometric Series</u>:

The sum of an infinite list of numbers (an infinite series) will do one of two things:

- Converge $\rightarrow$ Approach a specific value
- Diverge $\rightarrow$ Approach positive or negative infinity or bounce between 2 or more values

Ex) State whether the following series are convergent or divergent.

a) $-2+2-2+2-2+2-2+\ldots$

Bounces $\therefore$ divergent

b) $8+4+2+1+\frac{1}{2}+\frac{1}{4}+\ldots$ $\frac{1}{8}$ $\frac{1}{16}$

Approaches 0 $\therefore$ converges to 0

An infinite geometric series will converge if $-1 < r < 1$.

r has to be a decimal.

Proof: $$S_n = \frac{a(r^n - 1)}{r - 1}$$ ← sum of a geometric series

If r is between −1 and 1,

r^n gets smaller and approaches 0.

$$S_n = \frac{a(0-1)}{r-1}$$

$$= \frac{a(-1)}{r-1}$$

$$= \frac{-a}{r-1}$$

$$= \frac{a}{1-r}$$

Ex) If possible, find the sum of each series given below.

Not given t_n or n... infinite geometric.

a) $1 + \frac{1}{4} + \frac{1}{16} + \ldots$

$r = 1/4 = 0.25$ ∴ infinite converge

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{1}{1 - 1/4} \rightarrow \frac{1}{1} \times \frac{4}{3} =$$

$$\boxed{= \frac{4}{3}}$$

b) $9 + 3 + 1 + \frac{1}{3} + \ldots$

$\times \frac{1}{3}$

$r = \frac{1}{3}$ ∴ converges

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{9}{1 - 1/3}$$

$$\boxed{S_\infty = \frac{27}{2}}$$

c) $\frac{5}{3} + \frac{20}{9} + \frac{80}{27} + \ldots$

$4/3$

$r = \frac{4}{3} = 1.\overline{3}$

since $r > 1$, cannot find S_∞

d) $3 - \frac{3}{5} + \frac{3}{25} - \ldots$

$\times -\frac{1}{5}$

$r = -\frac{1}{5}$ ∴ converges

$$S_\infty = \frac{a}{1-r} = \frac{3}{1-(-1/5)} = \frac{5}{2}$$

Ex) Express $0.\overline{35}$ as a fraction.

0.353535...

r is between -1 and 1
So we can use infinite.

$$0.35 + 0.0035 + 0.000035 + \ldots$$

Series

$\times \frac{1}{100}$ $\times \frac{1}{100}$

$$S_\infty = \frac{a}{1-r} = \frac{0.35}{1-\frac{1}{100}} = \frac{35}{99}$$

Ex) Express $2.4\overline{21}$ as a fraction.

$$2.4 + \underbrace{0.021 \overset{\times \frac{1}{100}}{+} 0.00021 \overset{\times \frac{1}{100}}{+} 0.0000021 + \ldots}_{\text{Series}}$$

forget for now

$$S_\infty = \frac{a}{1-r} = \frac{7}{330}$$

This represents the series so now we have to add 2.4 back in.

$$2.4 + \frac{7}{330} = \frac{799}{330}$$

Now Try
Page 63 #1, 2, 3, 5, 6, 8, 9, 10, 11, 17