Unit 2 Systems of Equations & Inequalities

Review of Linear Systems of Equations:

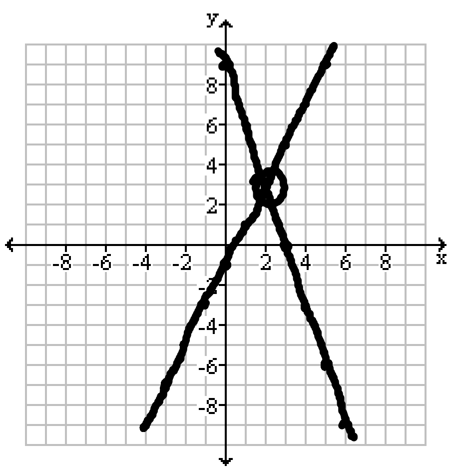
Systems of Equations:

A system of equations involves 2 or more equations that are considered at the same time.

Ex) Consider the system given by the following equations:

$$\begin{cases} y = 2x - 1 \\ y = -3x + 9 \end{cases}$$

> Both linear, straight lines $y = mx + b$



y int

- Graph $y = 2x - 1$ $\quad \frac{\text{rise}}{\text{run}} = \frac{2}{1}$
- Graph $y = -3x + 9$ $\quad \frac{\text{rise}}{\text{run}} = \frac{-3}{1}$

- Determine the solution to the system given by

$$y = 2x - 1$$
$$y = -3x + 9$$

$y = y$

$2x - 1 = -3x + 9$

Intersection Points

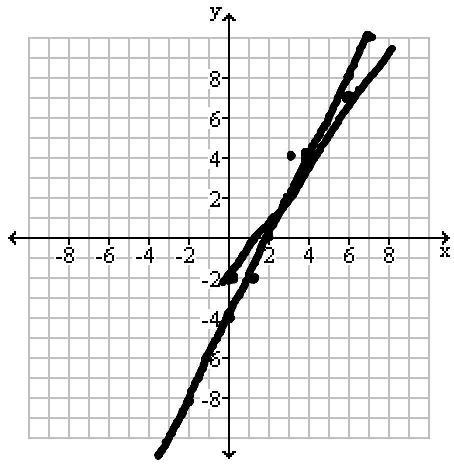
$x = 2$
$y = 3$ $\Big\}$ $(2, 3)$

Solving Systems Graphically:

Ex) Solve the following systems of equations by graphing each system.

① $y = 2x - 4$

$3x - 2y = 4 \longrightarrow \frac{\cancel{-3x}-2y}{-2} = \frac{-3x}{-2} + \frac{4}{-2} \rightarrow y = \frac{3}{2}x - 2$ ②



$m = \frac{\text{rise}}{\text{run}} = \frac{2}{1}$

$m = \frac{3}{2} = \frac{\text{rise}}{\text{run}}$

$x = 4$
$y = 4$ OR $(4, 4)$

Solving Systems by Graphing on the Calculator:

- Solve each equation for y.
- Enter equations into y_1 and y_2, then graph the equations. The point of intersection must be visible.
- Determine the point of intersection using the intersect feature.

Ex) Solve the following systems of equations by graphing (use your graphing calculator).

a) $2x+y=5 \rightarrow y=-2x+5$ ①

$x-2y=10 \rightarrow \frac{-2y}{-2}=\frac{-x}{-2}+\frac{10}{-2} \rightarrow y=\frac{1}{2}x-5$ ②

$x=4$
$y=-3$ OR $(4,-3)$

Slopes are negative reciprocals
∴ perpendicular slope

b) $5x+3y=6 \rightarrow \frac{3y}{3}=\frac{-5x}{3}+\frac{6}{3}$ ∴ ① $y=-\frac{5}{3}x+2$

$20x=24-12y$

$\rightarrow \frac{20x}{-12}-\frac{24}{-12}=\frac{-12y}{-12}$ ∴ $-\frac{5}{3}x+2=y$ ②

same

Slope is same ∴ parallel

Same line ∴ infinite solutions

Solving by Substitution:

- Solve for one variable in one of the equations (choose the one that is easiest to solve for).
- Substitute this expression into the other equation.
- Solve the single variable equation now created (solve for the 1st variable).
- Solve for the second variable by substituting the value already solved for into one of the equations.

Ex) Solve the following systems using the method of substitution.

a) $5x - 3y - 2 = 0$
$7x + y = 0 \rightarrow \boxed{y = -7x}$

Eqn 1: $5x - 3y - 2 = 0$

$5x - 3(-7x) - 2 = 0$

$5x + 21x - 2 = 0$

$26x - 2 = 0$

$\frac{\cancel{26}x}{\cancel{26}} = \frac{2}{26}$

$\boxed{x = \frac{2}{26} = \frac{1}{13}}$

$y = -7x$

$= -7\left(\frac{1}{13}\right)$

$\boxed{y = \frac{-7}{13}}$

$\left(\frac{1}{13}, \frac{-7}{13}\right)$

b) $x+4y=6$ $\quad x=-4y+6$
$2x-3y=1$

Eqn 2: $2x-3y=1$
$2(-4y+6)-3y=1$
$-8y+12-3y=1$
$-11y=1-12$
$\frac{\cancel{-11}y}{\cancel{-11}}=\frac{-11}{-11}$
$\boxed{y=1}$

$x=-4y+6$
$=-4(1)+6$
$\boxed{x=2}$

$(2,1)$

c) $-2x+y=9$
$5x+y+5=0$

Solving by Elimination:

- Arrange the equations so that one is above the other and corresponding terms and the "=" sign are aligned.
- Manipulate equations so the absolute value of the numerical coefficients of one pair of like terms are the same.
- Either add or subtract the equations to eliminate one of the variables.
- Solve for the 1st variable.
- Solve for the second variable by substituting the value already solved for into one of the equations.

Ex) Solve the following systems of equations using the method of elimination.

a) $3x + 2y = 19$, $\cancel{5x - 2y = 5}$

$$\begin{array}{rl} & 3x + 2y = 19 \\ + & 5x - 2y = 5 \\ \hline & \frac{\cancel{8}x}{\cancel{8}} = \frac{24}{8} \end{array}$$

$$\boxed{x = 3}$$

Sub $x=3$ into original:

$3(3) + 2y = 19$

$9 + 2y = 19$

$\frac{\cancel{2}y}{\cancel{2}} = \frac{10}{2}$

$$\boxed{y = 5}$$

b) $x + 7y = -25$
$5x + 13 = -7y$

Rearrange so we can compare.

$$x + 7y = -25$$
$$-\quad 5x + 7y = -13$$
$$\cancel{-4}x = -12$$
$$\frac{\cancel{-4}x}{\cancel{-4}} = \frac{-12}{-4}$$
$$\boxed{x = 3}$$

Sub $x=3$ into original

$$3 + 7y = -25$$
$$\frac{\cancel{7}y}{\cancel{7}} = \frac{-28}{7}$$
$$\boxed{y = -4}$$

c) $4x - 2y = -16$
$8x + 3y = -46$

$\times 2$: $8x - 4y = -32$
$$-\quad 8x + 3y = -46$$
$$-7y = 14$$

d) $\dfrac{x}{2} + \dfrac{4y}{3} = 4$
$3x + 5y = 6$

$\xrightarrow{\times 6}$ $\dfrac{6x}{2} + \dfrac{24y}{3} = 24 \rightarrow 3x + 8y = 24$
$$-\quad 3x + 5y = 6$$
$$3y = 18$$

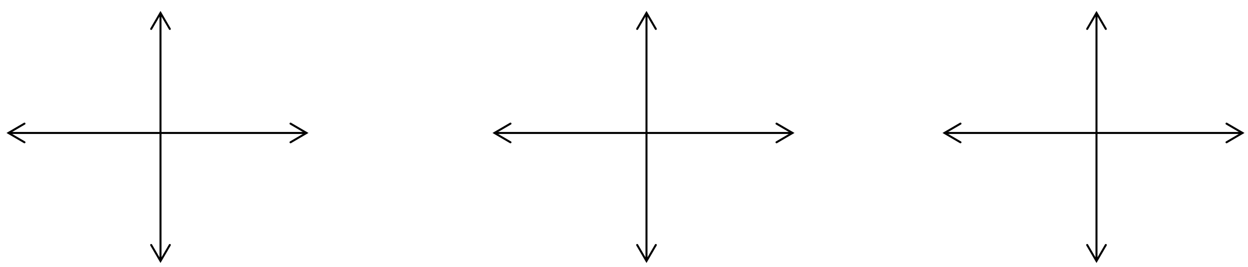
Now Try Worksheet

Non-Linear Systems:

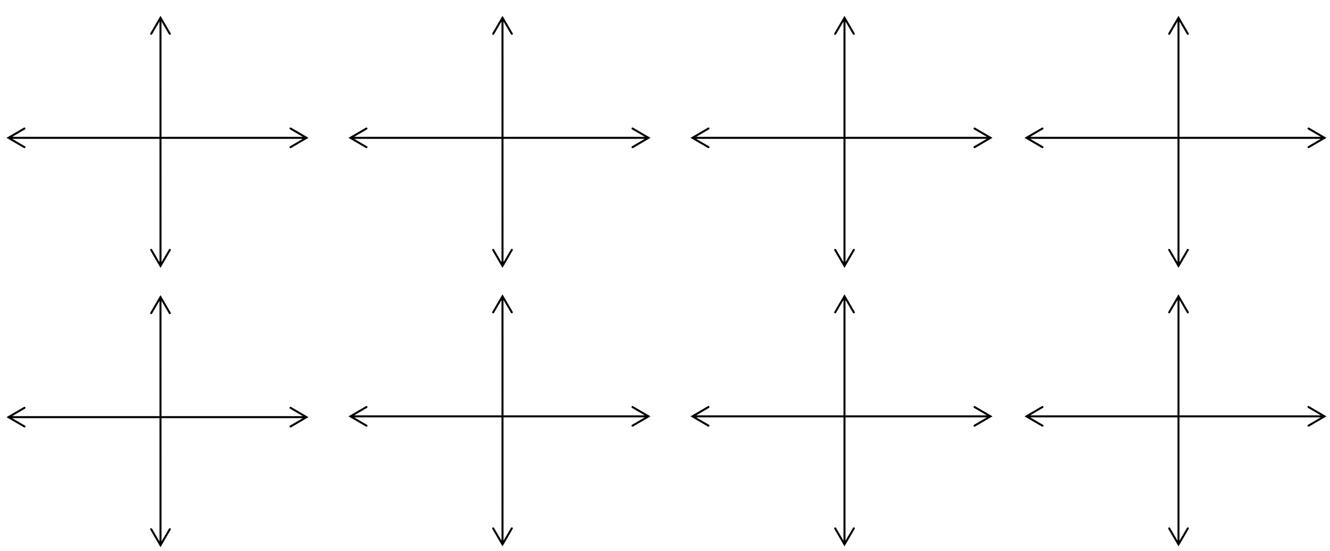
In this unit we will now consider systems of equations made up of a linear and a quadratic equation or systems made of two quadratic equations.

Possible Number of Solutions:

Linear-Quadratic



Quadratic-Quadratic



Steps for Solving Graphically:

- Solve each equation for *y*.
- Enter equations into y_1 and y_2, then graph the equations. The point(s) of intersection must be visible.
- Determine the point(s) of intersection using the intersect feature.

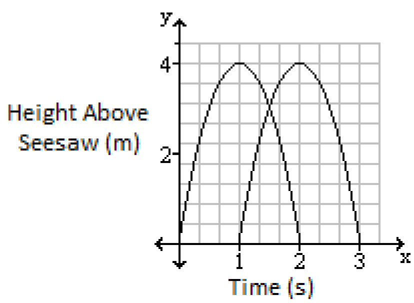
Ex) Solve the following systems graphically. Round to the nearest hundred if necessary.

a) $4x - y + 3 = 0$
$2x^2 + 8x - y + 3 = 0$

b) $2x^2 - 16x - y = -35$
$2x^2 - 8x - y = -11$

c) $y = -\tfrac{1}{3}(x-6)^2 + 18$
$y = -2(x-14)^2 + 16$

Ex) During a stunt, two Cirque du Soleil performers are launched toward each other from two slightly offset seesaws. The first performer is launched, and 1 second later the second performer is launched in the opposite direction. They both perform a flip and give each other a high five in the air. Each performer is in the air for 2 seconds. The height above the seesaw versus time for each performer during the stunt is approximated by a parabola as shown.



a) Determine the system of equation that models the performers' height during the stunt.

b) Solve the system graphically using your calculator.

c) Interpret your solution with respect to this situation.

Now Try
Page 435 #1, 2, 4, 5, 7, 9, 10, 12, 15,

Solving by Substitution:

Linear-Quadratic System:

- Solve for y in the linear equation.
- Substitute this expression into the quadratic equation.
- Solve the quadratic equation by combining like terms and using the method of factoring, completing the square, or quadratic formula.
- Solve for the y-variable by substituting the x-value(s) already solved for into one of the equations.

Quadratic-Quadratic System:

- Solve one of the equations for y.
- Substitute this expression in for y in the other equation.
- Solve the quadratic equation by combining like terms and using the method of factoring, completing the square, or quadratic formula.
- Solve for the y-variable by substituting the x-value(s) already solved for into one of the equations.

Ex) Solve the following systems using the method of substitution.

a) $5x - y = 10$ linear $\quad 5x-10=y$

$x^2 + x - 2y = 0$ quadratic

$x^2+x-2(5x-10)=0$

$x^2+x-10x+20=0$

$x^2-9x+20=0$

$(x-5)(x-4)=0$

$x=5 \qquad x=4$

When $x=5$,

$y=5(5)-10$

$=15$

When $x=4$;

$y=5(4)-10$

$=10$

$(5,15) \quad (4,10)$

b) $y = 3x^2 + 24x - 41$
$6x - y + 17 = 0$

y is already Solved for

$6x - (3x^2 + 24x - 41) + 17 = 0$

$6x - 3x^2 - 24x + 41 + 17 = 0$

$-3x^2 - 18x + 58 = 0$

UGLY → Solve Quad

$$x = \frac{18 \pm \sqrt{(-18)^2 - 4(-3)(58)}}{2(-3)}$$

$$= \frac{18 \pm \sqrt{1020}}{-6}$$

$x = -8.3229$ $\quad$ $x = 2.3229$

when $x = 2.3229..$

$y = 6x + 17$
$= 6(2.3229) + 17$
$= 30.9374$

When $x = -8.3229$
$y = 6(-8.3229) + 17$
$= -32.9374$

$(2.3229, 30.9374)$ $\quad$ $(-8.3229, -32.9374)$

c) $x^2 + 2y = 40$
$3x + y - 12 = 0$

$y = -3x + 12$

$x^2 + 2(-3x + 12) = 40$

$x^2 - 6x + 24 - 40 = 0$

$x^2 - 6x - 16 = 0$ I factored.

$(x - 8)(x + 2) = 0$

$x = 8$ $\quad$ $x = -2$

When $x = 8$;

$y = -3(8) + 12$
$= -24 + 12$
$= 12$

When $x = -2$;

$y = -3(-2) + 12$
$= 6 + 12$
$= 18$

$\boxed{(8, 12) \quad (-2, 18)}$

d) $3x^2 - x - y - 2 = 0$

$6x^2 + 4x - y = 4$ $\therefore y = 6x^2 + 4x - 4$

$3x^2 - x - (6x^2 + 4x - 4) - 2 = 0$

$3x^2 - x - 6x^2 - 4x + 4 - 2 = 0$

$-3x^2 - 5x + 2 = 0$

$x = \dfrac{5 \pm \sqrt{(-5)^2 - 4(-3)(2)}}{2(-3)}$

$= \dfrac{5 \pm \sqrt{49}}{-6}$

$x = \dfrac{5+7}{-6} = -2$ $\qquad x = \dfrac{5-7}{-6} = \dfrac{1}{3}$

When $x = -2$,

$y = 6(-2)^2 + 4(-2) - 4$

$= 24 - 8 - 4$

$= 12$

When $x = \frac{1}{3}$;

$y = 6\left(\frac{1}{3}\right)^2 + 4\left(\frac{1}{3}\right) - 4$

$= \frac{6}{9} + \frac{4}{3} - 4$

$= -2$

$\boxed{(-2, 12) \quad \left(\frac{1}{3}, -2\right)}$

e) $y = x^2 + 6x - 28$

$x^2 + 12x = -34 - 3y$

f) $4y+28=x^2-12x+36$

$y-65=2x^2-24x$ $\therefore y=2x^2-24x+65$

$4(2x^2-24x+65)+28=x^2-12x+36$

$8x^2-96x+260+28=x^2-12x+36$

$7x^2-84x+252=0$

$x=\frac{84\pm\sqrt{(-84)^2-4(7)(252)}}{2(7)}$

$=\frac{84\pm\sqrt{0}}{14}$ ← We will only have 1 solution.

$=\frac{84}{14}$

$=6$

When $x=6$;

$y=2(6)^2-24(6)+65$

$=72-144+65$

$=-7$

$\boxed{(6,-7)}$

Now Try
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Solving by Elimination:

- Arrange the equations so that one is above the other and corresponding terms and the "=" sign are aligned.
- Manipulate equations so the absolute value of the numerical coefficients on the y-term are the same.
- Either add or subtract the equations to eliminate the y-variables.
- Solve for the x-variable.
- Solve for the y-variable by substituting the value(s) of x already solved for into one of the equations.

Ex) Solve the following systems using the method of elimination.

a) $3x + 2y = 29$
$4y = x^2 - 10x + 13$

$2y = -3x + 29$ (×2)
$4y = x^2 - 10x + 13$

$4y = 0 - 6x + 58$
$-4y = x^2 - 10x + 13$

$0 = -x^2 + 4x + 45$ Solve Quad.

$x = \frac{-4 \pm \sqrt{(4)^2 - 4(-1)(45)}}{2(-1)}$

$= \frac{-4 \pm \sqrt{196}}{-2}$

$= \frac{-4 \pm 14}{-2}$

$x = -5$

$x = 9$

when $x = -5$;

$y = -\frac{3}{2}x + \frac{29}{2}$

$= -\frac{3}{2}(-5) + \frac{29}{2}$

$= 22$

When $x = 9$,

$y = -\frac{3}{2}(9) + \frac{29}{2}$

$= 1$

$\boxed{(-5, 22)(9, 1)}$

b) $y + 14 + x^2 = -14x \rightarrow y = -x^2 - 14x - 14$
$y = 5x + 76$

$$\begin{aligned} y &= -x^2 - 14x - 14 \\ -\ y &= 5x + 76 \\ \hline 0 &= -x^2 - 19x - 90 \\ &= -(x^2 + 19x + 90) \\ &= -(x+9)(x+10) \end{aligned}$$

factor

$\therefore x = -9 \quad x = -10$

When $x = -9$;

$y = 5(-9) + 76$
$= 31$

When $x = -10$;

$y = 5(-10) + 76$
$= 26$

$\boxed{(-9, 31)\ (-10, 26)}$

c) $4y - 5 = x^2 - 6x \rightarrow 4y = x^2 - 6x + 5$
$2y - 16 = 3x + 9 \qquad 2y = 3x + 25 \quad \times 2$

$$\begin{aligned} 4y &= x^2 - 6x + 5 \\ -\ 4y &= 6x + 50 \\ \hline 0 &= x^2 - 12x - 45 \\ 0 &= (x-15)(x+3) \end{aligned}$$

$\therefore x = 15 \quad x = -3$

When $x = -3$

$2y = 3(-3) + 25$
$2y = -9 + 25$
$\frac{2y}{\cancel{2}} = \frac{16}{2}$
$y = 8$

When $x = 15$

$2y = 3(15) + 25$
$2y = 45 + 25$
$\frac{2y}{\cancel{2}} = \frac{70}{2}$
$y = 35$

$\boxed{(-3, 8)\ (15, 35)}$

d) $3x^2 - 36x + y = -75$

$+ \quad 3x^2 - 54x - y = -189$

$6x^2 - 90x = -264$

$6x^2 - 90x + 264 = 0$

$6(x^2 - 15x + 44) = 0$

$6(x - 11)(x - 4) = 0$

$\therefore \; x = 11 \quad x = 4$

$\boxed{(11, -42) \; (4, 21)}$

When $x = 4$;

$3x^2 - 54x - y = -189$

$3(4)^2 - 54(4) - y = -189$

$48 - 216 - y = -189$

$48 - 216 + 189 = y$

$y = 21$

When $x = 11$

$3(11)^2 - 54(11) - y = -189$

$363 - 594 - y = -189$

$y = -42$

e) $6y = (x-7)^2 + 3 \rightarrow 6y = (x-7)(x-7) + 3$

$3y = -(x-13)^2 + 9 \rightarrow 3y = -(x-13)(x-13) + 9$

$\rightarrow 6y = x^2 - 7x - 7x + 49 + 3 \quad \rightarrow 6y = x^2 - 14x + 52$

$\rightarrow 3y = -(x^2 - 13x - 13x + 169) + 9 \rightarrow \underline{3y = -x^2 + 26x - 160}$

f) $4x - x^2 = 4y + 13$

$y + 20 = x^2 - 20x + 100$

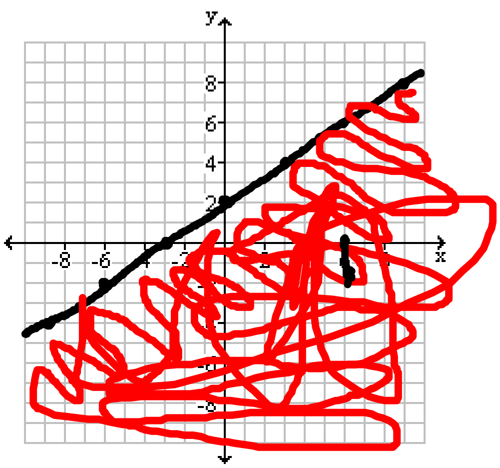
Now Try
Page 452 **#4, 6, 7, 10, 11, 13 (see example #3 on page 445 as a guide), 20**

Graphing Linear Inequalities in 2 Variables:

As we have seen linear inequalities in one variable can be graphed on a number line (one dimension). Linear inequalities in two variables must be graphed on a coordinate plane (two dimensions).

Ex) Graph the inequality given below.

$$y \le \frac{2}{3}x + 2$$



slope   y int

- Graph the line $y = \frac{2}{3}x + 2$
- Because we want $y \le \frac{2}{3}x + 2$ (y is less than $\frac{2}{3}x + 2$) shade the region below the line
- Check your answer (check with a point in the shaded region).

How do we distinguish between $y > 2x + 1$ and $y \ge 2x + 1$?

$\rightarrow y > 2x + 1$ is graphed with a dotted line ------------

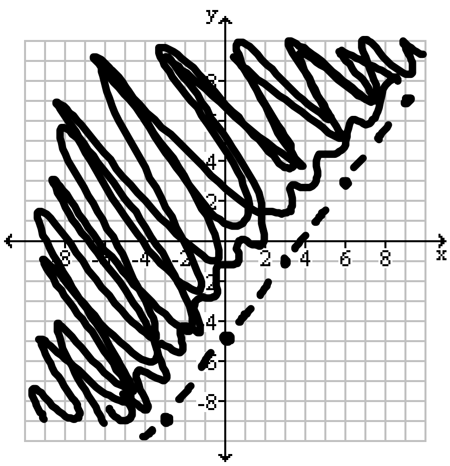
because we are not including points on line

$\rightarrow y \ge 2x + 1$ is graphed with a solid line ————

because every point on line is also in solution.

Ex) Graph each of the following inequalities. Check your solutions.

a) $4x - 3y + 12 < 27$



$4x - 3y + 12 < 27$

$-3y < -4x \underbrace{-12 + 27}$

$\frac{\cancel{-3}y}{\cancel{-3}} < \frac{-4x}{-3} + \frac{15}{-3}$

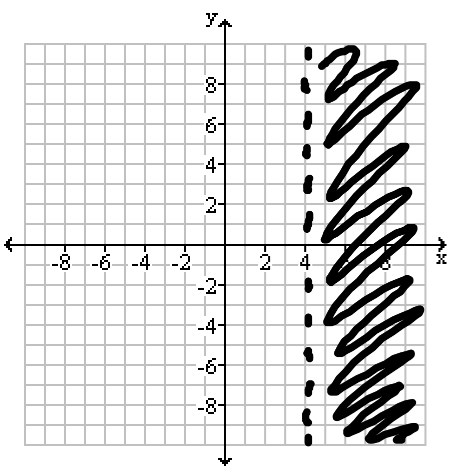
$y > \frac{4}{3}x - 5$

dotted slope y int

*When you divide by −, your inequality flips.

b) $x > 4$

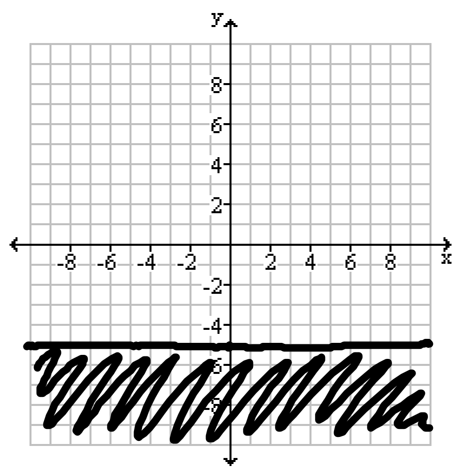
dotted



Graph $x = 4$

c) $y \le -5$

Solid



Graph $y = -5$

Inequalities with Restrictions:

Restrictions limit the region to be shaded.

Ex) Graph the following inequalities with restrictions.

a) $y \geq \frac{2}{3}x - 5$, $\left.\begin{matrix} x \geq 0 \\ y \leq 0 \end{matrix}\right\}$ restrictions

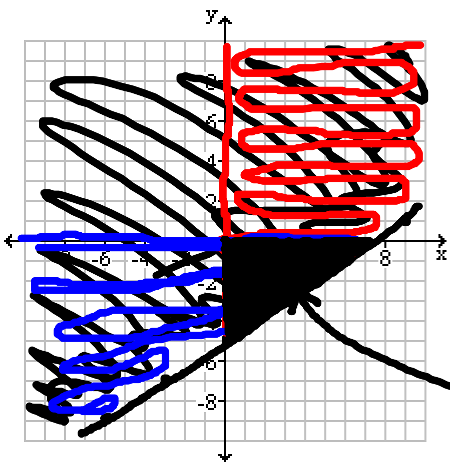
→ Graph as usual: $y = \frac{2}{3}x - 5$

slope y int

→ Shade above our line ($\geq$)

→ Shade restriction in different:

Solution is where all colors overlap

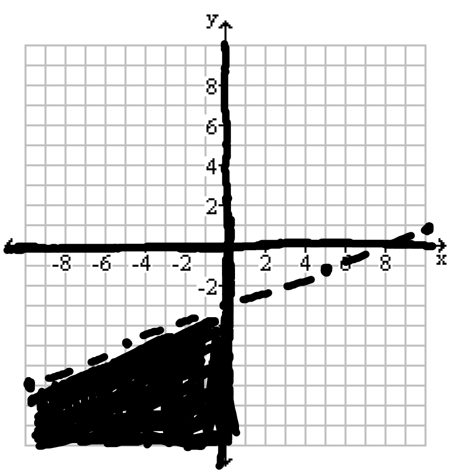


b) $2x - 5y > 15$, $\begin{matrix} x \leq 0 \\ y \leq 0 \end{matrix}$

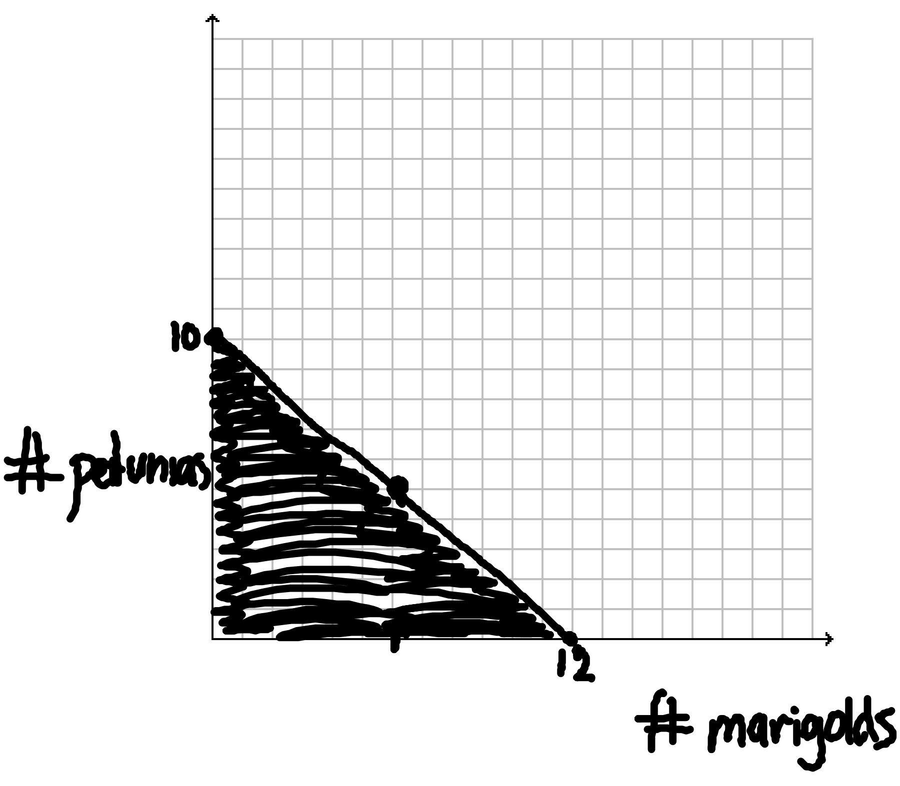
$y < \frac{2}{5}x - 3$

Slope yint

Graph using dotted



Ex) Mary is buying flowers for her yard. During a sale she can buy a flat o f marigolds for \$5 and a flat of petunias for \$6. If she has \$60 to spend, how many of each type of flower can she buy. Graph your results.



$5x + 6y \leq 60$

$\frac{\cancel{6}y}{\cancel{6}} \leq \frac{60}{6} - \frac{5x}{6}$

$y \leq \underset{\text{y-int}}{10} - \underset{\text{slope}}{\frac{5}{6}}x$

$x = $ # marigolds

$y = $ # petunias

Now Try
Page 472 #1, 4, 7, 8, 9, 11, 13, 17

<u>Solving Quadratic Inequalities with One Variable</u>:

Solving Linear Inequalities with One Variable (Review):

Ex) Solve the following.

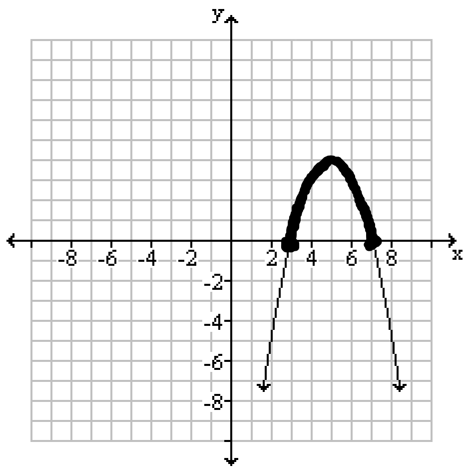
a) $5x-4=2x+23$ b) $-2y+7-3y=47$

Solving Quadratic Inequalities Graphically:

- Bring all terms to one side of the inequality
- Graph the related quadratic function and determine the desired regions
    - If function is less than 0, identify regions below the *x*-axis
    - If function is greater than 0, identify regions above the *x*-axis
    - If function can equal 0, include all points on the *x*-axis
- Make inequality statements that represent each region

Ex) Solve the inequality below given its graph.

Graph of $y = 10x - x^2 - 21$



$\underbrace{10x - x^2 - 21}_{Y_1} \geq 0$

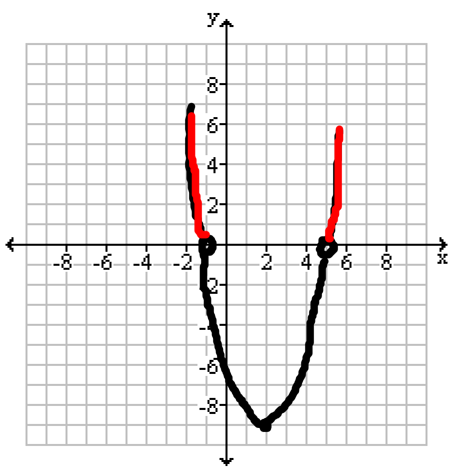
Where is this graph ≥ 0?

$\boxed{3 \leq x \leq 7}$

These x values are included in solution

Ex) Solve the following inequalities.

a) $(x-2)^2 > 9$



Graph: $y = (x-2)^2 - 9$

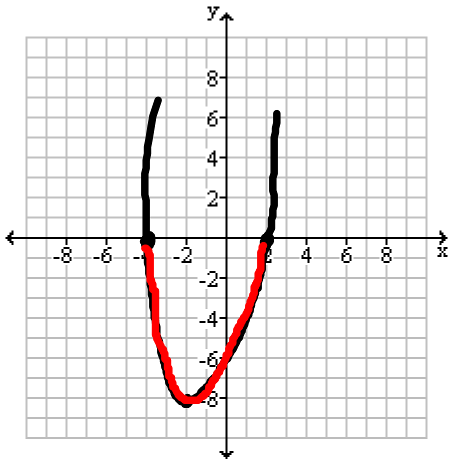
$(x-2)^2 - 9 > 0$

* You can't tell your x-int initially so find them graph, factor, square root principle, Quad formula.

Where is graph above 0?

$\boxed{x < -1,\ x > 5}$

b) $x^2 + 2x - 8 \le 0$



Graph

Vertex: $(-2,-8)$

x-int: x^2+2x-8

$(x+4)(x-2)$

$x=-4 \quad x=2$

Where is graph ≤ 0?

$\boxed{-4 \le x \le 2}$

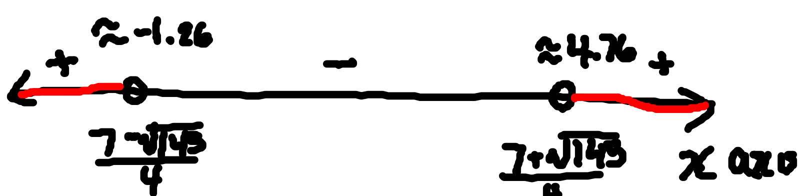
Solving Quadratic Inequalities Using Number Lines and Test Points:

- Bring all terms to one side of the inequality
- Find the roots (use the Quadratic Formula)
- Identify roots on a number line
- Use test points before, between, and after the roots
- Determine for which intervals the inequality works

Ex) Solve the following inequalities.

a) $2x^2 - 7x > 12$

$2x^2-7x-12>0$ no x int.



roots/x int:

$x = \frac{7 \pm \sqrt{(-7)^2-4(2)(-12)}}{2(2)}$

$= \frac{7\pm\sqrt{145}}{4}$ ← can't break this so leave it.

left: $x=-2$

$2(-2)^2-7(-2)-12=10$ so >0

Middle: $x=0$

$2(0)^2-7(0)-12=-12 < 0$

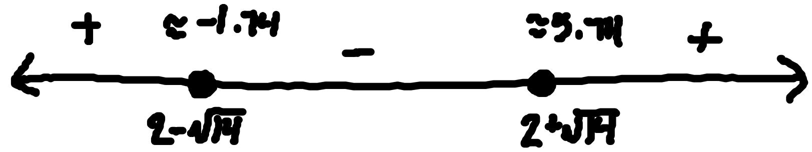
Right: 5

$2(5)^2-7(5)-12=3$ so >0

$$\boxed{x < \frac{7-\sqrt{145}}{4} \;,\quad x > \frac{7+\sqrt{145}}{4}}$$

b) $x^2 - 4x \le 10$

$x^2 - 4x - 10 \le 0$



$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-10)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{56}}{2}$$

$$= \frac{4 \pm 2\sqrt{14}}{2}$$

$$= 2 \pm \sqrt{14}$$

left: $x = -3$

$(-3)^2 - 4(-3) - 10 = 11 > 0$

Middle: $x = 0$

$0^2 - 4(0) - 10 = -10 < 0$

Right: $x = 7$

$(7)^2 - 4(7) - 10 = 11 > 0$

$\sqrt{56}$

$= 2\sqrt{14}$

$$\boxed{2-\sqrt{14} \le x \le 2+\sqrt{14}}$$

Now Try
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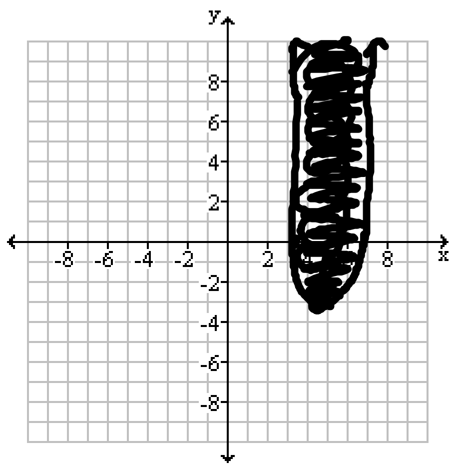
<u>Graphing Quadratic Inequalities with Two Variables</u>:

Like linear inequalities with two variables, we will solve quadratic inequalities with two variables by graphing a related equation and shading the appropriate region.

Ex) Graph the inequality given below.

$$y \geq (x-5)^2 - 3$$



- Graph the parabola
  $y = (x-5)^2 - 3$ Vertex: (5, -3)
- Because we want
  $y \geq (x-5)^2 - 3$
  (y is greater than $(x-5)^2 - 3$)
  shade the region above the parabola
- Check your answer (check with a point in the shaded region).

*Remember

if $y \leq$ or $y \geq$ use a solid line

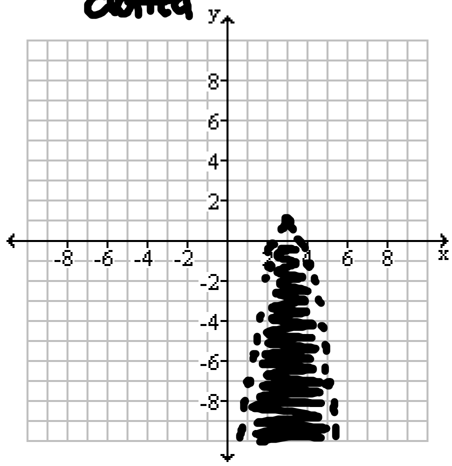
if $y <$ or $y >$ use a dotted line

Ex) Graph the following quadratic inequalities.

a) $y < -2(x-3)^2 + 1$

dotted

Vertex: (3,1)



*dotted line is not in our solution.

b) $y \leq x^2 - 2x - 3$

solid

Vertex.

$x^2 - 2x - 3$

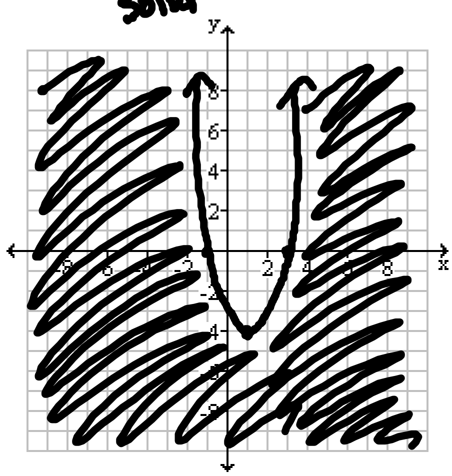
$x^2 - 2x + 1 - 1 - 3$

$(x-1)^2 - 4$

x-int. $x^2 - 2x - 3$

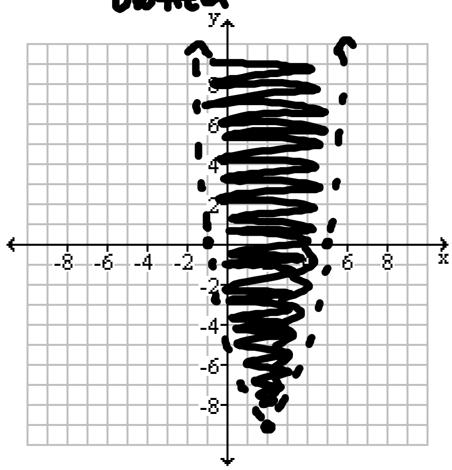
$(x-3)(x+1)$

$x = 3 \quad x = -1$



c) $y > x^2 - 4x - 5$

dotted



Vertex: $(2, -9)$

$x^2 - 4x + 4 - 4 - 5$

$(x-2)^2 - 9$

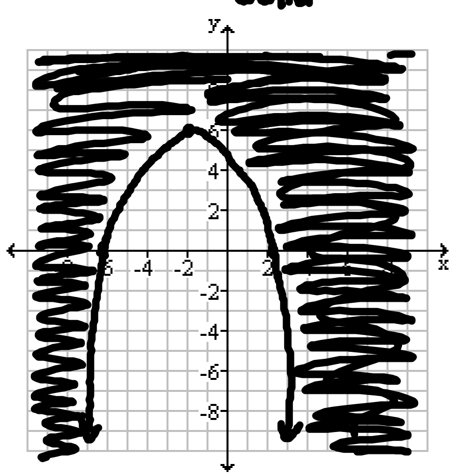
x-int: $x^2 - 4x - 5$

$(x-5)(x+1)$

$x = 5 \quad x = -1$

d) $x^2 + 3y \geq 14 - 4x$

solid



$\frac{\cancel{3}y}{\cancel{3}} \geq \frac{-x^2}{3} - \frac{4x}{3} + \frac{14}{3}$

$y \geq -\frac{x^2}{3} - \frac{4x}{3} + \frac{14}{3}$

$y \geq -\frac{1}{3}[x^2 + 4x - 14]$

$y \geq -\frac{1}{3}[x^2 + 4x + 4 - 4 - 14]$

$y \geq -\frac{1}{3}[(x+2)^2 - 18]$

$y \geq -\frac{1}{3}(x+2)^2 + 6$

Vertex: $(-2, 6)$

Now Try
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